1 Section 2.1 - 1, 2, 4, 5

1. If \( a, b \in \mathbb{R} \), prove the following

(a) If \( a + b = 0 \), then \( b = -a \).

Proof. By 2.1.1(A3), we start with \(-a = -a + 0\).

\[
\begin{align*}
-a &= -a + 0 = -a + (a + b) & \text{(assumption)} \\
    &= (-a + a) + b & \text{(2.1.1(A2))} \\
    &= 0 + b & \text{(2.1.1(A4))} \\
    &= b & \text{(2.1.1(A3))}
\end{align*}
\]

(b) \(-(-a) = a\)

Proof. By 2.1.1(A4), \( a + (-a) = 0 \) or \(-a + a = 0 \) (by A1). Then by part (a), \( a = -(-a) \).

(c) \((-1)a = -a\)

Proof. By 2.1.1(A4), \( a + (-a) = 0 \). We would like to show \( a + (-1)a \) also equals zero. To that end,

\[
\begin{align*}
a + (-1)a &= (1 \cdot a) + (-1 \cdot a) & \text{(2.1.1(M3))} \\
    &= (1 + -1)a & \text{(2.1.1(D))} \\
    &= 0 \cdot a & \text{(2.1.2(c))}
\end{align*}
\]

Since additive inverses are unique, \(-a = (-1)a\).

(d) \((-1)(-1) = 1\)

Proof. By (b) and (c) and letting \( a = -1 \), \((-1)(-1) = -(-1) = 1\), where the first equality is due to (c) and the second is due to (b).

2. Prove that if \( a, b \in \mathbb{R} \), then

(a) \(- (a + b) = (-a) + (-b)\)

Proof.

\[
\begin{align*}
-(a + b) &= (-1)(a + b) & \text{(Ex. 1, (a))} \\
        &= (-1)a + (-1)b & \text{(2.1.1(D))} \\
        &= -a + -b & \text{(Ex. 1, (c))}
\end{align*}
\]
(b) \((-a) \cdot (-b) = a \cdot b\)

**Proof.**

\[
(-a) \cdot (-b) = (-1)a \cdot (-1)b \quad \text{(Ex. 1, (c))}
\]
\[
= (-1)(-1)a \cdot b \quad \text{(2.1.1(M1))}
\]
\[
= 1a \cdot b \quad \text{(Ex. 1, (d))}
\]
\[
= a \cdot b \quad \text{(2.1.1(M3))}
\]

\(\square\)

(c) \(1/(-a) = -(1/a)\)

**Proof.** We want to show

i. \(-a \cdot 1/ - a = 1\)

ii. \(-a \cdot -(1/a) = 1\)

We get the first one directly by 2.1.1(M4). For the second part, \(-a \cdot -(1/a) = a \cdot 1/a = 1\), where the first equality is due to part (b).

\(\square\)

(d) \(-a/b = -a/b\) if \(b \neq 0\)

**Proof.**

\[
-(a/b) = -(a \cdot 1/b) \quad \text{(defn. of division)}
\]
\[
= -(a \cdot 1/b) \quad \text{(Ex. 1, (c))}
\]
\[
= ((-1) \cdot a) \cdot 1/b \quad \text{(2.1.1(M2))}
\]
\[
= -a \cdot 1/b \quad \text{(Ex. 1, (c))}
\]
\[
= -a/b \quad \text{(defn. of division)}
\]

\(\square\)

3. If \(a \in \mathbb{R}\) satisfies \(a \cdot a = a\), prove either \(a = 0\) or \(a = 1\).

**Proof.** If \(a = 0\), then \(a \cdot 0 = 0\) by 2.1.2(c). Otherwise, by 2.1.2(b), we get (since multiplicative identities are unique) that \(a \cdot a = a\) implies \(a = 1\).

Therefore, \(a = 0\) or \(a = 1\).

\(\square\)

4. If \(a \neq 0\) and \(b \neq 0\), show that \(1/(ab) = (1/a)(1/b)\).
Proof. We want to show $1 = (ab) \cdot 1/(ab)$ and $(ab) \cdot (1/a)(1/b) = 1$ which will prove the statement since multiplicative identities are unique. By 2.1.1(M4), we get $1 = (ab) \cdot 1/(ab)$. To get the other, note

$$
(ab) \cdot (1/a)(1/b) = (b \cdot a)(1/a) \cdot (1/b) \quad (2.1.1(M1)) \\
= b \cdot (a \cdot 1/a) \cdot (1/b) \quad (2.1.1(M2)) \\
= b \cdot 1 \cdot 1/b \quad (2.1.1(M4)) \\
= (b \cdot 1) \cdot 1/b \quad (2.1.1(M2)) \\
= b \cdot 1/b \quad (2.1.1(M3)) \\
= 1 \quad \text{(via similar argument)}
$$

\qed