1 Section 3.3 - 1, 2, 3, 4

1. **1. Let $x_1 := 8$ and $x_{n+1} := \frac{1}{2}x_n + 2$ for $n \in N$. Show that (x_n) is bounded and monotone. Find the limit.**

   Proof. First, let's show that it is monotone (decreasing). Note that $x_1 = 8 > x_2 = 6$. For induction, assume that $x_k > x_{k+1}$. Next notice

   $$\begin{aligned} x_k &> x_{k+1} \\ \frac{1}{2}x_k &> \frac{1}{2}x_{k+1} \\ \frac{1}{2}x_k + 2 &> \frac{1}{2}x_{k+1} + 2 \\ x_{k+1} &> x_{k+2} \end{aligned}$$

   Since the sequence is decreasing and $x_n > 0$ for all n, we can be assured that the upper bound is 8 and we have a lower bound of 0.

   To find the limit, we use the same technique as the examples in the book and class.

   $$\begin{aligned} x_{n+1} &= \frac{1}{2}x_n + 2 \\ x &= \frac{1}{2}x + 2 \qquad \text{both limits must be x} \\ x &= 4 \end{aligned}$$

   □

2. **2. Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in N$. Show that (x_n) is bounded and monotone. Find the limit.**

   Proof. Since $x_1 > 1$, we have that $1/x_1 < 1$. This means that $x_2 = 2 - 1/x_1 > 1$. We'd like to show that $x_n > 1$ for all n. By induction, we see that if $x_k > 1$, then $x_{k+1} = 2 - 1/x_k > 1$ (easy enough?). At this point we are bounded above by 2 and below by 1. (Note: We could have made another arguement that it was bounded below by 0. The bound of 1 is a little better, but the MCT does not require anything more than bounded.)

   Now let's show that it is monotone (decreasing). To do this, we want to start by looking at

$$
\begin{aligned}
0 &< (x_1 - 1)^2 \\
0 &< x_1^2 - 2x_1 + 1 \\
2x_1 &< x_1^2 + 1 \\
2 &< x_1 + 1/x_1 \\
2 - 1/x_1 &< x_1 \\
x_2 &< x_1
\end{aligned}
$$

Ack! That only takes care of the first case. Now we use induction to finish the monotone.

$$
\begin{aligned}
x_{k+1} &< x_k \\
1/x_{k+1} &> 1/x_k \\
-1/x_{k+1} &< -1/x_k \\
2 - 1/x_{k+1} &< 2 - 1/x_k \\
x_{k+2} &< x_{k+1}
\end{aligned}
$$

To find the limit, we use the same technique as the examples in the book and class.

$$
\begin{aligned}
x_{n+1} &= 2 - 1/x_n \\
x &= 2 - 1/x \\
x &= 1
\end{aligned}
$$

□

3. **3. Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in N$. Show that (x_n) is bounded below by 2 and monotone. Find the limit.**

Proof. Since $x_1 \geq 2$, we have that $x_1 - 1 \geq 1$.

$$
\begin{aligned}
x_1 - 1 &\geq 1 \\
\sqrt{x_1 - 1} &\geq 1 \\
1 + \sqrt{x_1 - 1} &\geq 2 \\
x_2 &\geq 2
\end{aligned}
$$

Using induction (and a similar argument), we find $x_{n+1} \geq 2$. That takes care of bounded. We now need to work on monotone. To do this we start by noticing that if $x_1 - 1 > 1$, then $x_1 - 1 > \sqrt{x_1 - 1} \rightarrow x_1 > 1 + \sqrt{x_1 - 1} = x_2$. That is our base case, we then assume that $x_k > x_{k+1}$

$$\begin{aligned} x_{k+1} &< x_k \\ x_{k+1} - 1 &< x_k - 1 \\ \sqrt{x_{k+1} - 1} &< \sqrt{x_k - 1} \\ 1 + \sqrt{x_{k+1} - 1} &< 1 + \sqrt{x_k - 1} \\ x_{k+2} &< x_{k+1} \end{aligned}$$

Again, we use the same technique as the examples in the book and class.

$$\begin{aligned} x_{n+1} &= 2 - 1/x_n \\ x &= 2 - 1/x \\ x &= 1 \end{aligned}$$

□

4. **4. Let $x_1 = 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in N$. Show that (x_n) converges and find the limit.**

Proof. Like the first problem, let's show that it is monotone (increasing) first. Note that $x_1 = 1 < x_2 = \sqrt{3}$. For induction, assume that $x_k < x_{k+1}$. Next notice

$$\begin{aligned} x_k &< x_{k+1} \\ 2 + x_k &< 2 + x_{k+1} \\ \sqrt{2 + x_k} &< \sqrt{2 + x_{k+1}} \\ x_{k+1} &< x_{k+2} \end{aligned}$$

Since the sequence is increasing and $x_n > 0$ for all n, we now want to get an upper bound on the sequence. To do this, note that $x_1 = 1 < x_2 = \sqrt{3} < 4$. Using induction that $x_k < 4$, we obtain

$$\begin{aligned} x_{k+1} &= \sqrt{2 + x_k} \\ &< \sqrt{2 + 4} \\ &< \sqrt{6} \\ &< 4 \end{aligned}$$

This again, allows us to use MCT to say that the sequence converges. Lastly, to find the limit, we do what we have done for the first 3 problems.

$$\begin{aligned} x_{n+1} &= \sqrt{2+x_n} \\ x &= \sqrt{2+x} \\ x &= 2 \end{aligned}$$

□