1 Section 3.2 - 1b, 1d, 3, 6b, 7, 19, 21

1. **1. For x_n given by the following formulas, establish either the convergence or the dievergence of the sequence $X = (x_n)$.**

   (a) $x_n := \frac{(-1)^n n}{n+1}$

   Proof. The even subsequence goes to 1, the odd subsequence to -1, hence the sequence diverges. □

   (b) $x_n := \frac{2n^2+1}{n^2+1}$

   Proof. Using the n^{th}-term test.

   $$\lim\left(\frac{2n^2+1}{n^2+1}\right) = \lim\left(\frac{2+1/n^2}{1+1/n^2}\right) = 2 \tag{1}$$

   Hence the sequence converges and converges to the value of 2. □

2. **3.**

   Show that if X and Y are sequences such X and X + Y are convergent, then Y is convergent.

   Proof. If we write $Y = (X+Y) - X$, we can use the fact that both $(X+Y)$ and X converge and the fact that the difference of two convergent sequences is also convergent. □

3. **6b. Find the limit of $\lim\left(\frac{(-1)^n}{n+2}\right)$.**

   Proof. We note that $-1/n < \frac{(-1)^n}{n+2} < 1/n$ for all $n \in N$. By the Squeeze Theorem this tells us that $\lim\left(\frac{(-1)^n}{n+2}\right) = 0$. □

4. **7. If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 CANNOT be used.**

   Proof. Let $K(\epsilon)$ be such that $1/K(\epsilon) < \epsilon/M$ and $|b_n| \leq M$ for all n.

   $$|a_n b_n - 0| = |a_n b_n| = |a_n|\,|b_n| \leq M\,|a_n| < M\epsilon/M = \epsilon \tag{2}$$

   whenever $n \geq K(\epsilon)$. □

5. **19. Let (x_n) be a sequence of positive real numbers such that $\lim(x_n^{1/n}) = L < 1$. Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in N$. Use thise to show that $\lim(x_n) = 0$.**

Proof. Since $L < 1$ there exists $r \in \Re$ such that $L < r < 1$. Note also, that since $L < 1$, $x_n^{1/n} < r$ for all sufficiently large $n \in N$. This yields $0 < x_n^{1/n} < r < 1$ or $0 < x_n < r^n < 1$.

To finish this off, we invoke the Squeeze Theorem and note that since $r < 1$ that $r^n \to 0$. $\square$

6. 21. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\epsilon > 0$ there exists M such that $|x_n - y_n| < \epsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?

Proof. We will show that $(y_n) \to x(\leftarrow x_n)$.

Given $\epsilon > 0$, choose K_1 so that $|y_n - x_n| < \epsilon/2$ whenever $n > K_1$. Likewise, choose K_2 so that $|x_n - x| < \epsilon/2$ whenever $n > K_2$. Then

$$\begin{aligned} |y_n - x| &= |y_n - x_n + x_n - x| \leq |y_n - x_n| + |x_n - x| && (3) \\ &< \epsilon/2 + \epsilon/2 = \epsilon && (4) \end{aligned}$$

Hence, $(y_n) \to x$. What this says is if we have two sequences (one of which converges) that stay as close as we want, then they both converge to the same value. $\square$