1 Section 3.1 - 1d, 2b, 3b, 4, 5a, 5c, 8, 11

1. **1d. Write the first five terms of $x := \frac{1}{n^2+2}$.**

   Proof. $x_1 = \frac{1}{2}, x_2 = \frac{1}{5}, x_3 = \frac{1}{10}$ $x_4 = \frac{1}{17}, x_5 = \frac{1}{26}$ □

2. **2b. The first few terms of a sequence (x_n) are given below. Give a formula for the n^{th} term x_n.** $1/2, -1/4, 1/8, -1/16, \ldots$.

   $x_n = \frac{(-1)^{n+1}}{2^n}$

3. **3b. List the five terms of the following inductively defined sequences $y_1 := 2, y_{n+1} = \frac{1}{2}(y_n + 2/y_n)$.**

   Proof. $y_1 = 2, y_2 = \frac{3}{2}, y_3 = \frac{17}{12}$ $y_4 = \frac{577}{408}, y_5 = \frac{665857}{470832}$ □

4. **4. For any $b \in \Re$, prove that $\lim(b/n) = 0$.**

   Proof. Let $K(\epsilon)$ be such that $1/K(\epsilon) < \epsilon/|b|$. Then

   $$|\frac{b}{n} - 0| = |b| * \frac{1}{n} < |b| * 1/K(\epsilon) < |b| * \epsilon/|b| < \epsilon \quad (1)$$

   The above line works whenever $n \geq K(\epsilon)$. □

5. **5. Use the definition of the limit of a sequence to establish the folliwng limits.**

   (a) $\lim \frac{n}{n^2+1} = 0$

   Proof. We want to show that given an $\epsilon > 0$, we can find a $K(\epsilon)$ that satisfies $|\frac{n}{n^2+1} - 0| < \epsilon$ for all $n \geq K(\epsilon)$. Our first job is to find $K(\epsilon)$ and then we have to show it works (satisfies the definition).
   Part I: Finding $K(\epsilon)$ - If we look at $|\frac{n}{n^2+1} - 0| < \epsilon$, we get $|\frac{n}{n^2+1}| < \epsilon$. We recognize that $\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$. At this point, we know there exists a $K(\epsilon)$ such that $1/K(\epsilon) < \epsilon$ whenever $n \geq K(\epsilon)$.
   Part II: Making sure $K(\epsilon)$ works! (Note: This is the important step!) We start by looking at $|\frac{n}{n^2+1} - 0| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \epsilon$ whenever $n \geq K(\epsilon)$. □

   (b) $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$

   Proof. For this one, I'll skip Part I. What we need for Part II though is a $K(\epsilon)$ such that the definition works. Let $K(\epsilon)$ be such that $\frac{1}{K(\epsilon)} < \frac{4}{13}\epsilon$.
   We start by looking at $|\frac{3n+1}{2n+5} - \frac{3}{2}| = |\frac{6n+2-6n-15}{4n+10}| < |\frac{-13}{4n}| = \frac{13}{4} * \frac{1}{n} < \frac{13}{4} * \frac{4}{13}\epsilon < \epsilon$ whenever $n \geq K(\epsilon)$.

   □

6. 8. Prove that $\lim(x_n) = 0$ **if and only if** $\lim(|x_n|) = 0$**. Give an example to show that the convergence of** $(|x_n|)$ **need not imply the convergence of** (x_n)**.**

Proof. If $\lim(x_n) = 0$, then there exists a $K(\epsilon)$ such that $|x_n| < \epsilon$ whenever $n \geq K(\epsilon)$. And since we have the following set of equalities: $|x_n| = ||x_n|| = ||x_n| - 0|$, we are able to say that $\lim(|x_n|) = 0$. Likewise, if we follow the inequalities in the other direction, we get the other direction of the statment.

For the example (or counterexample), use $\frac{(-1)^n n}{n+1}$. □

7. 11.

Show that $\lim\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$**.**

Proof. Let $K(\epsilon)$ be such that $1/K(\epsilon) < \epsilon$.

$|\frac{1}{n} - \frac{1}{n+1} - 0| = |\frac{n+1-n}{n(n+1)}| = \frac{1}{n^2+n} < \frac{1}{n^2} < \frac{1}{n} < \epsilon$ whenever $n \geq K(\epsilon)$. □