1 Section 2.4 - $\sup$ portion of 4(a,b), and 6, 13, 18

1. **4.**

   (a) **Let** $a > 0$ **and let** $aS := \{as : s \in S\}$**. Prove** $\sup(aS) = a \sup S$**.**

   Proof. Let $u = \sup(S)$. This means that u is an upper bound of S. Hence, $s \leq u$ for all s. Since $a > 0$, $as \leq au$ for all s. This means au is an upper bound of aS, so $\sup(aS) \leq au = a\sup(S)$.

   For the other direction, let $v = \sup(aS)$. Again, we get $as \leq v$ for all s. Since $a > 0$, we can divide both side to obtain $s \leq v/a$ for all s. This implies that v/a is an upper bound on S or that $\sup(S) \leq v/a$ or rather $a\sup(S) \leq v = \sup(aS)$.

   Put the two inequalities together to get the necessary result. ☐

   (b) **Let** $b < 0$ **and let** $bS := \{bs : s \in S\}$**. Prove** $\sup(bS) = b \inf S$**.**
   Left to reader (as it is a mirror of (a)).

2. **6. Let A and B be bounded nonempty subsets of** $\Re$**, and let** $A+B := \{a + b : a \in A \textbf{and} b \in B\}$**. Prove that** $\sup(A + B) = \sup(A) + \sup(B)$**.**

   There are two ways of proving this (1) showing inequalities in both directions or (2) straight equality. We will do the second:

   Proof. We will use the fact that the sets are bounded and non-empty along with the fact that $\sup(a+S) = a+\sup(S)$ (proved in a previous problem).

   Then let $u = \sup A$ and $v = \sup B$ and choose $a \in A$ as a fixed element. Then $\sup(a + B) = a + \sup(B) = a + v$. Now, $\sup(A + B) = \{a + b : a \in A \text{ and } b \in B\} = \{a + v : a \in A\} = \{a : a \in A\} + v = \sup(A) + v = u + v$. ☐

3. **13. If** $y > 0$**, show that there exists** $n \in N$ **such that** $1/2^n < y$**.**

   Proof. By Corollary 2.4.5, we know there exists an $n \in N$ such that $1/n < y$. Since we didn't do this particular problem, it is necessary to show that $n < 2^n$. You would do this by induction (easy proof). Because of that fact, we see that $1/2^n < 1/n$ and we have the necessary conclusion. ☐

4. **18. If** $u > 0$ **is any real number and** $x < y$**, show that there exists a rational number** r **such that** $x < ru < y$**.**

   Proof. Since $u > 0$ and $x < y$, then we know that $\frac{x}{u} < \frac{y}{u}$. By the Density Theorem, we know there exists $r \in Q$ such that $\frac{x}{u} < r < \frac{y}{u}$. We finish the proof by multiplying the inequalities by u. ☐