1 Section 2.2 - 1(b), 4, 6(a), 15, 16(a)

1. If $a, b \in \Re$ and $b \neq 0$. Show $|a/b| = |a| / |b|$

   Proof.

   $$\begin{aligned} |a/b| &= |a \cdot 1/b| && \text{defn. of division} \\ &= |a| \cdot |1/b| && 2.2.2(a) \end{aligned}$$

   Now the question we must examine is: Does $|1/b| = 1/|b|$?

   The answer is yes and is justified by using the definition on $|1/b|$, then using Exercise 2.1.2(c) to obtain the right side of the equation. □

2. Show that $|x - a| < \epsilon$ iff $a - \epsilon < x < a + \epsilon$.

   Proof. ($\Rightarrow$) If $|x - a| < \epsilon$, then by 2.2.2(c)

   $$-\epsilon < x - a < \epsilon \tag{1}$$

   Then by adding a to everything (2.1.7(b)),

   $$a - \epsilon < x < a + \epsilon \tag{2}$$

   .

   ($\Leftarrow$) If $a - \epsilon < x < a + \epsilon$, subtract a from everything (2.1.7(b)).

   $$-\epsilon < x - a < \epsilon \tag{3}$$

   Then by 2.2.2(c), we have $|x - a| < \epsilon$.

   □

3. Find all $x \in \Re$ that satisfies $|4x - 15| \leq 13$.

   Solution is the interval $[1/2, 9/2]$.

4. Show that if $a, b \in \Re$ and $a \neq b$, then there exists ϵ-neighborhoods of U of a and V of b such that $U \cap V = \emptyset$.

   Proof. If $a \neq b$, then $|a - b| \neq 0$. Define $\epsilon := \frac{|a-b|}{2}$. Then define

   $$\begin{aligned} U_\epsilon(a) &:= \left\{ x \in \Re : |x - a| < \frac{|a-b|}{2} \right\} && \text{and} \\ V_\epsilon(b) &:= \left\{ x \in \Re : |x - b| < \frac{|a-b|}{2} \right\} && \text{and} \end{aligned}$$

   If you draw the picture of these sets you will see that the intersection is indeed empty. □

5. Show that if $a, b \in \Re$, then

(a) $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$

Proof. WLOG assume $a > b$. This means $|a-b| = a-b$.

$$\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+a-b) = a = \max\{a,b\} \tag{4}$$

If $b > a$, proof follows the same except $|a-b| = b-a$.

□

(b) $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$

Proof. WLOG assume $a > b$. This means $|a-b| = a-b$.

$$\frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b-(a-b)) = b = \min\{a,b\} \tag{5}$$

If $b > a$, proof follows the same except $|a-b| = b-a$.

□