1 Section 2.1 - 10, 13, 14, 20, 22

1. (a) If $a < b$ and $c < d$, prove that $a + c < b + d$.

   Proof. We will show two cases:

   i. $c < d$
   To show this, if $a < b$, then by 2.1.6(a), $b - a \in P$. Likewise, if $c < d$, then $d - c \in P$. By 2.1.5(i), $(b - a)+)d - c) \in P$. This implies $(b + d) - (a + c) \in P$, which by definition 2.1.6(a) means $b + d > a + c$.

   ii. $c = d$
   By 2.1.7(b), if $a < c$ and $c = d$, we have $a + c < b + c$ or $a + c < b + d$.

   □

   (b) If $0 < a < b$ and $0 \leq c \leq d$, prove $0 \leq ac \leq bd$.

   Proof. By 2.1.7(c), if $0 < a < b$ then $c \cdot 0 \leq ac \leq bc$ or $0 \leq ac \leq bc$. We need the $\leq$ because c could equal zero. To finish, by 2.1.7(c) and $c \leq d$, $0 \leq ac \leq bd \leq db$ or $0 \leq ac \leq bd$. □

2. If $a, b \in \Re$, show $a^2 + b^2 = 0$ iff $a = 0$ and $b = 0$.

   Proof. ($\Leftarrow$) Assume $a = 0$ and $b = 0$, then $(0)^2 + (0)^2 = 0 + 0 = 0$
   ($\Rightarrow$) Now assume $a^2 + b^2 = 0$ and $a \neq 0$ and $b \neq 0$. This implies $a^2 = -b^2$. By 2.1.8(a), $a^2 \in P$. Therefore, $-b^2 \in P$ or that $b^2 \notin P$. But this contradicts 2.1.8(a) unless both a and b are zero. □

3. (a) If $0 \leq a < b$, show that $a^2 \leq ab < b^2$.

   Proof. If $0 \leq a < b$, then multiplying through by a yields $a^2 \leq ab$. The inequality is a result of the case that $a = 0$. So $a^2 \leq ab < b \cdot b = b^2$ (by 2.1.7(c)). □

   (b) Show by example that it does not follow that $a^2 < ab < b^2$.
   If you let $a = 0$, then we have $0^2 < 0 < b^2$ and we have a false statement since $0 \not< 0$.

4. (a) If $0 < c < 1$, show that $0 < c^2 < c < 1$.

   Proof.

   $$\begin{aligned} 1 &> c && \text{by assumption} \\ &= c \cdot 1 && \text{2.1.1(M3)} \\ &> c \cdot c && \text{assumption and 2.1.7(c)} \\ &= c^2 && \\ &> 0 && \text{2.1.8(a)} \end{aligned}$$

   Therefore, $0 < c^2 < c < 1$. □

(b) If $1 < c$, show that $1 < c < c^2$.

Proof.

$$\begin{aligned} 1 &< c && \text{by assumption} \\ &= c \cdot 1 && \text{2.1.1(M3)} \\ &< c \cdot c && \text{assumption and 2.1.7(c)} \\ &= c^2 \end{aligned}$$

Therefore, $1 < c < c^2$. □

5. (a) If $c > 1$, show $c^n \geq c \ \forall n \in N$ and $c^n > c$ for $n > 1$.

   Proof. This is an induction proof where you have shown the first case in Ex. 20(b). So assume true for $n = k$ ($c^k \geq c$) and prove true for $n = k + 1$. If $c^k \geq c \Rightarrow c \leq c^k = 1 \cdot c^k \leq c \cdot c^k = c^{k+1}$. The last statement is due to the fact that $n = 1$ is not included. □

   (b) If $0 < c < 1$, show that $c^n \leq c \ \forall n \in N$ and that $c^n < c$ for $n > 1$.
   I leave this proof to the reader since it follows exactly the same way as the previous part.