1 Section 2.1 - 10, 13, 14, 20, 22

1. (a) If \( a < b \) and \( c < d \), prove that \( a + c < b + d \).

   Proof. We will show two cases:

   i. \( c < d \)

      To show this, if \( a < b \), then by 2.1.6(a), \( b - a \in P \). Likewise, if \( c < d \), then \( d - c \in P \). By 2.1.5(i), \((b - a) + (d - c) \in P \). This implies \((b + d) - (a + c) \in P \), which by definition 2.1.6(a) means \( b + d > a + c \).

   ii. \( c = d \)

      By 2.1.7(b), if \( a < c \) and \( c = d \), we have \( a + c < b + c \) or \( a + c < b + d \).

(b) If \( 0 < a < b \) and \( 0 \leq c \leq d \), prove \( 0 \leq ac \leq bd \).

   Proof. By 2.1.7(c), if \( 0 < a < b \) then \( c \cdot 0 \leq ac \leq bc \) or \( 0 \leq ac \leq bc \). We need the \( \leq \) because \( c \) could equal zero. To finish, by 2.1.7(c) and \( c \leq d \), \( 0 \leq ac \leq bd \leq db \) or \( 0 \leq ac \leq bd \).

2. If \( a, b \in \mathbb{R} \), show \( a^2 + b^2 = 0 \) iff \( a = 0 \) and \( b = 0 \).

   Proof. (\( \Leftarrow \)) Assume \( a = 0 \) and \( b = 0 \), then \((0)^2 + (0)^2 = 0 + 0 = 0 \)

   (\( \Rightarrow \)) Now assume \( a^2 + b^2 = 0 \) and \( a \neq 0 \) and \( b \neq 0 \). This implies \( a^2 = -b^2 \).

   By 2.1.8(a), \( a^2 \in P \). Therefore, \(-b^2 \in P \) or that \( b^2 \notin P \). But this contradicts 2.1.8(a) unless both \( a \) and \( b \) are zero.

3. (a) If \( 0 \leq a < b \), show that \( a^2 \leq ab < b^2 \).

   Proof. If \( 0 \leq a < b \), then multiplying through by \( a \) yields \( a^2 \leq ab \).

   The inequality is a result of the case that \( a = 0 \). So \( a^2 \leq ab < b^2 \) (by 2.1.7(c)).

   (b) Show by example that it does not follow that \( a^2 < ab < b^2 \).

   If you let \( a = 0 \), then we have \( 0^2 < 0 < b^2 \) and we have a false statement since \( 0 \not< 0 \).

4. (a) If \( 0 < c < 1 \), show that \( 0 < c^2 < c < 1 \).

   Proof.

   \[
   \begin{align*}
   1 & > c & \text{by assumption} \\
   = c \cdot 1 & \quad \text{2.1.1(M3)} \\
   > c \cdot c & \quad \text{assumption and 2.1.7(c)} \\
   = c^2 & \quad \text{2.1.8(a)} \\
   > 0 & \\
   \end{align*}
   \]

   Therefore, \( 0 < c^2 < c < 1 \).
(b) If $1 < c$, show that $1 < c < c^2$.

Proof.

$$
\begin{align*}
1 < c & \quad \text{by assumption} \\
= c \cdot 1 & \quad \text{2.1.1(M3)} \\
< c \cdot c & \quad \text{assumption and 2.1.7(c)} \\
= c^2
\end{align*}
$$

Therefore, $1 < c < c^2$. \hfill \Box

5. (a) If $c > 1$, show $c^n \geq c \forall n \in N$ and $c^n > c$ for $n > 1$.

Proof. This is an induction proof where you have shown the first case in Ex. 20(b). So assume true for $n = k$ ($c^k \geq c$) and prove true for $n = k + 1$. If $c^k \geq c \Rightarrow c \leq c^k = 1 \cdot c^k \leq c \cdot c^k = c^{k+1}$. The last statement is due to the fact that $n = 1$ is not included. \hfill \Box

(b) If $0 < c < 1$, show that $c^n \leq c \forall n \in N$ and that $c^n < c$ for $n > 1$.

I leave this proof to the reader since it follows exactly the same way as the previous part.