1 Section 2.1 - 8 // Section 1.2 - 1, 3, 4, 7, 20

1. Show if $x, y \in Q$, then

   (a) $x + y$ is rational and

   (b) xy is rational.

   Proof. If $x, y \in Q$, then let $x = p/q$ and $y = r/s$.

   i. $x + y = \frac{p}{q} + \frac{r}{s} = \frac{ps+rq}{qs}$. The question now is whether $\frac{ps+rq}{qs}$ is rational. The answer is yes. Since the integers are closed under multiplication and addition, $a = ps + rq \in Z$ and $b = qs \in Z$. Hence, $x + y = \frac{a}{b} \in Q$.

   ii. $xy = \frac{p}{q}\frac{r}{s} = \frac{pr}{qs}$. Again, for similar reasons we can conclude that $xy \in Q$.

   □

   (c) i. Prove if $x \in Q$ and $y \in \Re - Q$, then $x + y \in \Re - Q$.

   ii. If, in addition, $x \neq 0$, then show $xy \in \Re - Q$.

   Proof. i. By what was done in (a)(i), if $x + y \in Q$ we have that $x + y = \frac{p}{q} + \frac{r}{s}$ which are both in Q. But we know that y cannot be written as a fraction, hence a contradiction that $x + y \in Q$.

   ii. Similarly, assume that $xy \in Q$ and using 8(a), we see that it forces $y \in Q$ (another contradiction).

   □

1. Prove that $1/(1 * 2) + 1/(2 * 3) + \cdots + 1/(n * (n + 1)) = n/(n + 1)$ for all $n \in N$.

   Proof. (a) Show true for $n = 1$: $\frac{1}{1\cdot(1+2)} = \frac{1}{2}$. CHECK.

   (b) Assume true for $n = k$ and show true for $n = k + 1$.
   Begin with $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$. Add $\frac{1}{(k+1)(k+2)}$ to both sides. By simplifying the right side you are left with $\frac{k+1}{k+2}$ which is exactly what is needed on the right.

   □

2. Prove that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all $n \in N$.

   Proof. (a) Show true for $n = 1$: $3 = 4(1)^2 - 1$. CHECK.

(b) Assume true for $n = k$ and show true for $n = k + 1$.
As with the previous problem, start with $n = k$ or $3 + 11 + \cdots + (8k - 5) = 4k^2 - k$ and add $8(k + 1) - 5$ to both sides. Working the right side,

$$\begin{aligned} 4k^2 - k + (8(k + 1) - 5) &= 4k^2 - k + 8k + 8 - 5 \\ &= 4k^2 + 7k + 3 \end{aligned}$$

Unfortunately, this doesn't look like $4(k+1)^2 - (k+1)$ (or does it?). By expanding that quantity, we, in fact, do come up with the other statement. Hence the result holds.

□

3. Prove $1^2 + 3^2 + \cdots + (2n - 1)^2 = (4n^3 - n)/3$ for all $n \in N$.

Proof. Same game as the previous two. Add $(2(k + 1) - 1)^2$ to both sides of the $n = k$ case and simplify. □

4. Prove that $5^{2n} - 1$ is divisible by 8 for all $n \in N$.

Proof. This one is a little different than previous problems, but we attack it the same way.

(a) Show true for $n = 1$: $5^{2 \cdot 1} - 1 = 25 - 1 = 24 = 8 \cdot 3$. CHECK.

(b) Assume true for $n = k$ and show true for $n = k + 1$.
If true for $n = k$, this means that $5^{2k} - 1 = 8 \cdot m_1$. To show true for $n = k + 1$ means, we have to show $5^{2(k+1)} - 1 = 8 \cdot m_2$.

$$\begin{aligned} 5^{2(k+1)} = 5^{2k+2} - 1 &= 25 \cdot 5^{2k} - 1 \\ &= 25 \cdot 5^{2k} - 1 - 24 + 24 \\ &= 25 \cdot 5^{2k} - 25 + 24 \\ &= 25(5^{2k} - 1) + 24 \end{aligned}$$

We know (by induction hypothesis) that $5^{2k} - 1$ is divisible by 8 and so is 24. Hence, since each term in the sum is divisible by 8, the sum is also divisible by 8.

□

5. Let the numbers x_n be defined as follows: $x_1 = 1, x_2 = 2$ and $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for all $n \in N$. Use the Principle of Strong Induction to show $1 \leq x_n \leq 2$ for all $n \in N$.

Proof. For this one, the trick is to figure out what S is. It is defined as $S := x_k : 1 \leq x_k \leq 2$defined by the recursive relation

(a) $1 \in S$ (Base case)

(b) Assume $x_k \in S$ and hence $x_{k-1} \in S$. We need to show $x_{k+1} \in S$.

$$x_{k+1} = (1/2)(x_k + x_{k-1}) \tag{1}$$

By the recursive relation combined with the induction step:

$$1 \leq x_k \leq 2$$
$$1 \leq x_{k-1} \leq 2$$

Adding those two inqualities together yields $2 \leq (x_k + x + k - 1) \leq 4$. By manipulating that inequality into $1 \leq \frac{1}{2}(x_k + x_{k-1}) \leq 2$, we come to the conclusion. (Inside the inequalities IS x_{k+1}.)

□