# 1 Section 2.1 - 1, 2, 4, 5

1. If $a, b \in \Re$, prove the following

   (a) If $a + b = 0$, then $b = -a$.

   *Proof.* By 2.1.1(A3), we start with $-a = -a + 0$.

$$\begin{aligned} -a = -a + 0 &= -a + (a + b) && \text{(assumption)} \\ &= (-a + a) + b && \text{(2.1.1(A2))} \\ &= 0 + b && \text{(2.1.1(A4))} \\ &= b && \text{(2.1.1(A3))} \end{aligned}$$

   □

   (b) $-(-a) = a$

   *Proof.* By 2.1.1(A4), $a + (-a) = 0$ or $-a + a = 0$ (by A1). Then by part (a), $a = -(-a)$. □

   (c) $(-1)a = -a$

   *Proof.* By 2.1.1(A4), $a + -a = 0$. We would like to show $a + (-1)a$ also equals zero. To that end,

$$\begin{aligned} a + (-1)a &= (1 \cdot a) + (-1 \cdot a) && \text{(2.1.1(M3))} \\ &= (1 + -1)a && \text{(2.1.1(D))} \\ &= 0 \cdot a && \text{(2.1.2(c))} \end{aligned}$$

   Since additive inverses are unique, $-a = (-1)a$.

   □

   (d) $(-1)(-1) = 1$

   *Proof.* By (b) and (c) and letting $a = -1$, $(-1)(-1) = -(-1) = 1$, where the first equality is due to (c) and the second is due to (b). □

2. Prove that if $a, b \in \Re$, then

   (a) $-(a + b) = (-a) + (-b)$

   *Proof.*

$$\begin{aligned} -(a + b) &= (-1)(a + b) && \text{(Ex. 1, (a))} \\ &= (-1)a + (-1)b && \text{(2.1.1(D))} \\ &= -a + -b && \text{(Ex. 1, (c))} \end{aligned}$$

   □

(b) $(-a) \cdot (-b) = a \cdot b$

*Proof.*

$$\begin{aligned}
(-a) \cdot (-b) &= (-1)a \cdot (-1)b && \text{(Ex. 1, (c))} \\
&= (-1)(-1)a \cdot b && \text{(2.1.1(M1))} \\
&= 1a \cdot b && \text{(Ex. 1, (d))} \\
&= a \cdot b && \text{(2.1.1(M3))}
\end{aligned}$$

□

(c) $1/(-a) = -(1/a)$

*Proof.* We want to show

i. $-a \cdot 1/-a = 1$

ii. $-a \cdot -(1/a) = 1$

We get the first one directly by 2.1.1(M4). For the second part, $-a \cdot -(1/a) = a \cdot 1/a = 1$, where the first equality is due to part (b). □

(d) $-(a/b) = -a/b$ if $b \neq 0$

*Proof.*

$$\begin{aligned}
-(a/b) &= (-(a \cdot 1/b) && \text{(defn. of division)} \\
&= -(a \cdot 1/b) && \text{(Ex. 1, (c))} \\
&= ((-1) \cdot a) \cdot 1/b && \text{(2.1.1(M2))} \\
&= -a \cdot 1/b && \text{(Ex. 1, (c))} \\
&= -a/b && \text{(defn. of division)}
\end{aligned}$$

□

3. If $a \in \Re$ satisfies $a \cdot a = a$, prove either $a = 0$ or $a = 1$.

   *Proof.* If $a = 0$, then $a \cdot 0 = 0$ by 2.1.2(c). Otherwise, by 2.1.2(b), we get (since multiplicative identities are unique) that $a \cdot a = a$ implies $a = 1$.

   Therefore, $a = 0$ or $a = 1$. □

4. If $a \neq 0$ and $b \neq 0$, show that $1/(ab) = (1/a)(1/b)$.

*Proof.* We want to show $1 = (ab) \cdot 1/(ab)$ and $(ab) \cdot (1/a)(1/b) = 1$ which will prove the statement since multiplicative identities are unique. By 2.1.1(M4), we get $1 = (ab) \cdot 1/(ab)$. To get the other, note

$$\begin{aligned}
(ab) \cdot (1/a)(1/b) &= (b \cdot a)(1/a) \cdot (1/b) && \text{(2.1.1(M1))} \\
&= b \cdot (a \cdot 1/a) \cdot 1/b && \text{(2.1.1(M2))} \\
&= b \cdot 1 \cdot 1/b && \text{(2.1.1(M4))} \\
&= (b \cdot 1) \cdot 1/b && \text{(2.1.1(M2))} \\
&= b \cdot 1/b && \text{(2.1.1(M3))} \\
&= 1 && \text{(via similar argument)}
\end{aligned}$$

□